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Applying Shannon's Function to entropy differences of arithmetic operations with triangular fuzzy numbers

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ABSTRACT

The main purposes of this paper are to probe for the entropy differences between arithmetically manipulated Triangular Fuzzy Numbers (TFNs) using Shannon's Function; and to study the relationships between any two TFNs. Simultaneously, we expand on Wang and Chiu's article [9]. The entropy differences between two TFNs subjected to different arithmetic operations are classified as fifteen theorems, and fifteen examples are used to prove the accuracy of these theorems. This paper finds that with the application of Shannon's Function on triangular fuzzy numbers the grades of fuzziness are changed after arithmetic operations.

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1. INTRODUCTION

In the past, researchers have used myriads of methods to measure the fuzziness of fuzzy sets. Kaufmann [4] denoted the fuzziness of a fuzzy set by calculating the distance between the fuzzy set and its nearest non-fuzzy set. De Luca et al. [5] proposed using the entropy to describe the fuzziness of a fuzzy set. Indeed, there are plenty more papers which discuss the entropy of fuzzy sets [3,4,8]. Pedrycz [6] used the fuzzy set with a triangular membership function to demonstrate the entropy change when the interval size of the universal set is changed. Wang and Chiu [9] extended the results of Pedrycz's findings to any type of fuzzy sets. It is necessary to base the comparisons of the fuzziness of several fuzzy numbers on the same definition of entropy computation.

In the past, the study of the entropy differences of TFNs with arithmetic operations has always relied on this equation of entropy functions: $h(x)=4x(1-x)$. Shannon's famous function, $h(x)=-x\cdot\ln x-(1-x)\cdot\ln(1-x)$, is often used to measure fuzziness, yet it is difficult to find researchers who use the function to calculate entropy differences. In this paper, we use Shannon's Function on arithmetic operations of triangular fuzzy numbers and to examine the entropy differences after the calculations. Moreover, we extend the studies of Wang et al. (2000), and Wang et al. (2005), which, using $h(x)=4x(1-x)$, found the entropy differences of triangular fuzzy numbers submitted to arithmetic operations.

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This paper contains five sections. In Section 2, the fuzzy sets and the properties of entropy are introduced. In Section 3, we use Shannon's Function to probe the entropy differences of arithmetic operations with TFNs, and prove fourteen theorems. In Section 4, several examples are given to illustrate the theorems. Finally, Section 5 briefly summarizes our conclusions.

2. ENTROPY AND FUZZY SETS

2.1 The properties of entropy

Suppose $\tilde{A}$ is a fuzzy set and is defined in a universal set U, where U is finite and real. The membership function value of x in fuzzy set $\tilde{A}$ for $x \in U$ is represented as $\tilde{A}(x)$ and $\tilde{A}(x): x \to [0,1]$. The measurement of fuzziness of the fuzzy set $\tilde{A}$ is denoted as $H(\tilde{A})$ and it holds the following properties (de Luca and Termini, 1972 [5]; Zimmermann, 1996 [11]):

1. $H(\tilde{A})=0$, if $\tilde{A}$ is a crisp set in U.
2. If $\tilde{A}(x)=1/2, \quad \forall x \in U$, then $H(\tilde{A})$ has a unique maximum.
3. For two fuzzy sets $\tilde{A}$ and $\tilde{B}$, if $\tilde{B}(x) \le \tilde{A}(x)$ for $\tilde{A}(x) \le 1/2$ and $\tilde{B}(x) \ge \tilde{A}(x)$ for $\tilde{A}(x) \ge 1/2$ then $H(\tilde{A}) \ge H(\tilde{B})$.
4. $H(\tilde{A}^C) = H(\tilde{A})$, where $\tilde{A}^C(x)$ is the standard complement of $\tilde{A}(x)$, that is, $\tilde{A}^C(x) = 1 - \tilde{A}(x)$.

The researchers use the integration of the following equation to calculate the *global entropy* measure of the fuzzy set $\tilde{A}$ independent of x over the universal set U [6].

$$H(\tilde{A}) = \int_{x \in X} h(\tilde{A}(x))p(x)dx \tag{1}$$

where $p(x)$ is the probability density function of the available data in U, $h(\tilde{A}(x))$ is the entropy function, $h(x):[0,1] \to [0,1]$ is monotonically increasing in $\left[0,\frac{1}{2}\right]$ and monotonically decreasing in $\left[\frac{1}{2},1\right]$, $h(x)=0$, as $x=0$ and $x=1$; $h(x)=1$, as $x=1/2$. The following equations are the well-known entropy functions used to measure the fuzziness of $H(\tilde{A})$ which can be regarded as an "entropy" of a fuzzy set $\tilde{A}(x)$:

$$h(x) = \begin{cases} 2x & if \quad x \in \left[0,\frac{1}{2}\right] \\ 2(1-x) & if \quad x \in \left[\frac{1}{2},1\right] \end{cases} \tag{2}$$

$$h(x) = 4x(1-x) \tag{3}$$

and

$$h(x) = -x \cdot \ln x - (1-x) \cdot \ln(1-x) \tag{4}$$

where Eq.(4) is called Shannon's function [11] , which is applied to calculate the entropy of the TFN in this paper.

2.2 Definitions of fuzzy sets

Definition 1. The Triangular Fuzzy Number (TFN) $\tilde{A}$ is denoted by a triplet (a_1, a_2, a_3), it is usual to represent this TFN as $\tilde{A} = (a_1, a_2, a_3)$. The membership function can be defined as follows [6, 11]:

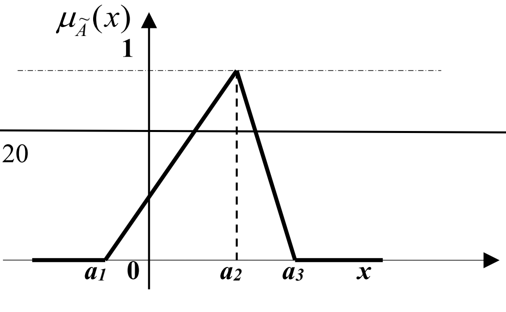

Figure 1: Triangular Fuzzy Number (TFN) $\tilde{A} = (a_1, a_2, a_3)$

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 \quad , & x < a_1, \;\; a_3 < x \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 \le x \le a_3 \end{cases} \tag{5}$$

where $\mu_{\tilde{A}}(x)$ is the degree of membership or the membership function value of x in fuzzy set $\tilde{A}$, U and $\mu_{\tilde{A}}(x)$ represent a universal set and membership function, respectively. In the expression below $\mu_{\tilde{A}}(x)$ and fuzzy set $\tilde{A}$ have the form: $\mu_{\tilde{A}}(x): U \to [0,1]$.

Definition 2. Two TFNs $\tilde{A}$ and $\tilde{B}$ are defined as $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$, so [6]

(i) Addition:

$$\tilde{A} \oplus \tilde{B} = (a_1, a_2, a_3) \oplus (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

(ii) Subtraction:

$$\tilde{A} - \tilde{B} = (a_1, a_2, a_3) - (b_1, b_2, b_3) = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$$

(iii) Multiplication:

(1) $n\tilde{A} = (na_1, na_2, na_3)$ $\quad n \in N \quad N$: denotes a natural number

(2) $m\tilde{A} = (ma_3, ma_2, ma_1)$ $\quad m \in I, m < 0 \quad I$: denotes an integer

(3) $r\tilde{A} = (ra_1, ra_2, ra_3)$ $\quad r \in R, r \ge 0 \quad R$: denotes a real number

(4) $s\tilde{A} = (sa_3, sa_2, sa_1)$ $\quad s \in R, s < 0$

$n\tilde{A}$, $m\tilde{A}$, $r\tilde{A}$, and $s\tilde{A}$ are triangular fuzzy numbers.

(iv) $\tilde{A}^{-}$ is the image of triangular fuzzy number $\tilde{A}$, defined as:

$$\tilde{A}^{-} = -(\tilde{A}) = (-a_3, -a_2, -a_1)$$

The results are still triangular fuzzy numbers through the arithmetic operations of addition to, subtraction from and multiplication by two triangular fuzzy numbers.

3. The entropy differences on TFNs through arithmetic operations

We will discuss and prove the entropy differences of triangular fuzzy numbers (TFNs) through arithmetic operations employing Eq.(4) above in this section. According to Definition 2, the result of putting two TFNs through arithmetic operations is always a TFN.

Suppose that we have two triangular fuzzy numbers $\tilde{A}$ and $\tilde{B}$, briefly indicated as $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$, respectively. Their membership functions are shown below:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 \quad , & x < a_1, \;\; a_3 < x \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 \le x \le a_3 \end{cases} \tag{6}$$

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$$\mu_{\widetilde{B}}(x)=\begin{cases}0 \quad , & x<b_1,\ b_3<x\\ \dfrac{x-b_1}{b_2-b_1}, & b_1\le x\le b_2\\ \dfrac{b_3-x}{b_3-b_2}, & b_2\le x\le b_3\end{cases} \tag{7}$$

Theorem 1. *For two TFNs* $\widetilde{A}$ *and* $\widetilde{B}$ *, expressed as* $\widetilde{A}=(a_1,a_2,a_3)$ *and* $\widetilde{B}=(b_1,b_2,b_3)$*, supposing* $p(x)=s$*, where s is a constant, then the relationship of the entropy of* $\widetilde{A}$ *and* $\widetilde{B}$ *has :* $H(\widetilde{A})=kH(\widetilde{B})$*, where* $k=\dfrac{2a_2-a_1-a_3}{2b_2-b_1-b_3}$.

Proof. Apply Eq.(4) to Eq.(6), we obtain,

$$h(\mu_{\widetilde{A}}(x))=\begin{cases}0 & , \quad x<a_1,\ a_3<x\\ -\left(\dfrac{x-a_1}{a_2-a_1}\right)\ln\left(\dfrac{x-a_1}{a_2-a_1}\right)-\left(1-\dfrac{x-a_1}{a_2-a_1}\right)\ln\left(1-\dfrac{x-a_1}{a_2-a_1}\right), & a_1\le x\le a_2\\ -\left(\dfrac{a_3-x}{a_3-a_2}\right)\ln\left(\dfrac{a_3-x}{a_3-a_2}\right)-\left(1-\dfrac{a_3-x}{a_3-a_2}\right)\ln\left(1-\dfrac{a_3-x}{a_3-a_2}\right), & a_2\le x\le a_3\end{cases}$$

Apply Eq.(1) to the above equation, and computing the entropy of TFN $\widetilde{A}$, the integration result is shown as,

$$H(\widetilde{A})=\int_{a_1}^{a_2}\left[-\left(\frac{x-a_1}{a_2-a_1}\right)\ln\left(\frac{x-a_1}{a_2-a_1}\right)-\left(1-\frac{x-a_1}{a_2-a_1}\right)\ln\left(1-\frac{x-a_1}{a_2-a_1}\right)\right]p(x)dx$$
$$+\int_{a_2}^{a_3}\left[-\left(\frac{a_3-x}{a_3-a_2}\right)\ln\left(\frac{a_3-x}{a_3-a_2}\right)-\left(1-\frac{a_3-x}{a_3-a_2}\right)\ln\left(1-\frac{a_3-x}{a_3-a_2}\right)\right]p(x)dx$$

Let $y=\dfrac{x-a_1}{a_2-a_1}$, then $dx=(a_2-a_1)dy$. $\quad z=\dfrac{a_3-x}{a_3-a_2},\quad dx=-(a_3-a_2)dz$.

$$H(\widetilde{A})=(-s)(a_2-a_1)\int_0^1[y\ln y+(1-y)\ln(1-y)]dy+(s)(a_3-a_2)\int_0^1[z\ln z+(1-z)\ln(1-z)]dz$$
$$=(-s)(a_2-a_1)\left[\int_0^1 y\ln y dy+\int_0^1(1-y)\ln(1-y)dy\right]+(s)(a_3-a_2)\left[\int_0^1 z\ln z dz+\int_0^1(1-z)\ln(1-z)dz\right]$$
$$=(-s)(a_2-a_1)\left[\left(\frac{1}{2}y^2\ln y-\frac{y^2}{4}\right)\Bigg|_0^1+\left(\frac{(1-y)^2}{4}-\frac{(1-y)^2}{2}\ln(1-y)\right)\Bigg|_0^1\right]$$
$$+(s)(a_3-a_2)\left[\left(\frac{1}{2}z^2\ln z-\frac{z^2}{4}\right)\Bigg|_0^1+\left(\frac{(1-z)^2}{4}-\frac{(1-z)^2}{2}\ln(1-z)\right)\Bigg|_0^1\right]$$
$$=(-s)(a_2-a_1)\left[\left(\frac{1}{2}\ln 1-\frac{1}{4}-0\right)+\left(0-\frac{1}{4}+\frac{1}{2}\ln 1\right)\right]$$
$$+(s)(a_3-a_2)\left[\left(\frac{1}{2}\ln 1-\frac{1}{4}-0\right)+\left(0-\frac{1}{4}+\frac{1}{2}\ln 1\right)\right]$$
$$=(-s)(a_2-a_1)\left(\ln 1-\frac{1}{2}\right)+(s)(a_3-a_2)\left(\ln 1-\frac{1}{2}\right)=s(a_1+a_3-2a_2)\left(\ln 1-\frac{1}{2}\right)$$
$$=\frac{s}{2}(2a_2-a_1-a_3)$$

Hence,
$$H(\widetilde{A})=\frac{s}{2}(2a_2-a_1-a_3)\cdot \tag{8}$$

Using the same method to integrate the entropy of $\widetilde{B}$, the result is

$$H(\tilde{B}) = \frac{s}{2}\left(2b_2 - b_1 - b_3\right) \tag{9}$$

$$\therefore \ \frac{H(\tilde{A})}{H(\tilde{B})} = \frac{2a_2 - a_1 - a_3}{2b_2 - b_1 - b_3} \ \Rightarrow H(\tilde{A}) = kH(\tilde{B}),\ where\ \ k = \frac{2a_2 - a_1 - a_3}{2b_2 - b_1 - b_3}.$$

Lemma 1. *For the two TFNs $\tilde{A}$ and $\tilde{B}$, if the following conditions are satisfied, then* $H(\tilde{A}) = H(\tilde{B})$.

(1) When $\begin{cases} a_1 = a_2 \\ b_1 = b_2 \end{cases}$, that is $\tilde{A}(x)$ and $\tilde{B}(x)$ are left-facing right triangles and they have the same "*base width*".

(2) When $\begin{cases} a_3 = a_2 \\ b_3 = b_2 \end{cases}$, that is $\tilde{A}(x)$ and $\tilde{B}(x)$ are right-facing right triangles and they have the same "*base width*".

(3) When $\begin{cases} a_1 + a_3 = 2a_2 \\ b_1 + b_3 = 2b_2 \end{cases}$, that is $\tilde{A}(x)$ and $\tilde{B}(x)$ are the isosceles triangles.

(4) When $2a_2 - a_1 - a_3 = 2b_2 - b_1 - b_3$.

Theorem 2. *Supposing $\tilde{C} = \tilde{A} + \tilde{B}$, $\tilde{C}$ being a TFN, then the entropy of $\tilde{C}$ is the summation of the entropies of $\tilde{A}$ and $\tilde{B}$. That is, $H(\tilde{C}) = H(\tilde{A} + \tilde{B}) = H(\tilde{A}) + H(\tilde{B})$.*

Proof. For $\tilde{C}(x) = \tilde{A}(x) + \tilde{B}(x)$, the membership function can be expressed as

$$\mu_{\tilde{C}}(x) = \begin{cases} 0 & , \quad x < a_1 + b_1,\ \ a_3 + b_3 < x \\ \dfrac{x - (a_1 + b_1)}{(a_2 + b_2) - (a_1 + b_1)}, & a_1 + b_1 \le x \le a_2 + b_2 \\ \dfrac{(a_3 + b_3) - x}{(a_3 + b_3) - (a_2 + b_2)}, & a_2 + b_2 \le x \le a_3 + b_3 \end{cases}$$

Let $y = \dfrac{x - (a_1 + b_1)}{(a_2 + b_2) - (a_1 + b_1)}$, $\quad z = \dfrac{(a_3 + b_3) - x}{(a_3 + b_3) - (a_2 + b_2)}$.

Applying Eq.(4) to the above equation, we obtain

$$h(\mu_{\tilde{C}}(x)) = \begin{cases} 0 & , \quad x < a_1 + b_1,\ \ a_3 + b_3 < x \\ -(y)\ln(y) - (1 - y)\ln(1 - y), & a_1 + b_1 \le x \le a_2 + b_2 \\ -(z)\ln(z) - (1 - z)\ln(1 - z), & a_2 + b_2 \le x \le a_3 + b_3 \end{cases}$$

Applying Eq.(1) to the above equation, and computing the entropy of TFN $\tilde{C}$, the integration result is shown as

$$H(\tilde{C}) = \int_{a_1+b_1}^{a_2+b_2} \left[-(y)\ln(y) - (1 - y)\ln(1 - y)\right] p(x)dx + \int_{a_2+b_2}^{a_3+b_3} \left[-(z)\ln(z) - (1 - z)\ln(1 - z)\right] p(x)dx$$

$$\because \ \ y = \frac{x - (a_1 + b_1)}{(a_2 + b_2) - (a_1 + b_1)} \qquad \therefore \ \ dx = ((a_2 + b_2) - (a_1 + b_1))dy$$

$$\because \ \ z = \frac{(a_3 + b_3) - x}{(a_3 + b_3) - (a_2 + b_2)} \qquad \therefore \ \ dx = -((a_3 + b_3) - (a_2 + b_2))dz$$

$$\begin{aligned} H(\tilde{C}) = (-s)\left[(a_2 + b_2) - (a_1 + b_1)\right] \int_0^1 \left[y\ln y + (1 - y)\ln(1 - y)\right] dy \\ + (s)\left[(a_3 + b_3) - (a_2 + b_2)\right] \int_0^1 \left[z\ln z + (1 - z)\ln(1 - z)\right] dz \end{aligned}$$

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$$=(-s)\left[(a_2+b_2)-(a_1+b_1)\right]\left(\ln 1-\frac{1}{2}\right)+(s)\left[(a_3+b_3)-(a_2+b_2)\right]\left(\ln 1-\frac{1}{2}\right)$$

$$=\frac{s}{2}\left(2(a_2+b_2)-(a_1+b_1)-(a_3+b_3)\right)$$

$$=\frac{s}{2}\left(2a_2-a_1-a_3\right)+\frac{s}{2}\left(2b_2-b_1-b_3\right)$$

$$=H(\tilde{A})+H(\tilde{B})$$

Hence, $H(\tilde{C})=\frac{s}{2}\left(2a_2-a_1-a_3\right)+\frac{s}{2}\left(2b_2-b_1-b_3\right)=H(\tilde{A})+H(\tilde{B})$.

Theorem 3. *Supposing* $\tilde{C}=\tilde{A}-\tilde{B}$ *,* $\tilde{C}$ *being a TFN, then the entropy of* $\tilde{C}$ *is the difference of the entropies of* $\tilde{A}$ *and* $\tilde{B}$ *. That is,* $H(\tilde{C})=H(\tilde{A}-\tilde{B})=H(\tilde{A})-H(\tilde{B})$.

Proof. For $\tilde{C}(x)=\tilde{A}(x)-\tilde{B}(x)$, the membership function can be expressed as

$$\mu_{\tilde{C}}(x)=\begin{cases} 0 & , \quad x<a_1-b_3, \ \ a_3-b_1<x \\ \dfrac{x-(a_1-b_3)}{(a_2-b_2)-(a_1-b_3)}, & a_1-b_3\le x\le a_2-b_2 \\ \dfrac{(a_3-b_1)-x}{(a_3-b_1)-(a_2-b_2)}, & a_2-b_2\le x\le a_3-b_1 \end{cases}$$

Let $y=\dfrac{x-(a_1-b_3)}{(a_2-b_2)-(a_1-b_3)}$, $\quad z=\dfrac{(a_3-b_1)-x}{(a_3-b_1)-(a_2-b_2)}$.

Applying Eq.(4) to the above equation, we obtain

$$h(\mu_{\tilde{C}}(x))=\begin{cases} 0 & , \quad x<a_1-b_3, \ \ a_3-b_1<x \\ -(y)\ln(y)-(1-y)\ln(1-y), & a_1-b_3\le x\le a_2-b_2 \\ -(z)\ln(z)-(1-z)\ln(1-z), & a_2-b_2\le x\le a_3-b_1 \end{cases}$$

Applying Eq.(1) to the above equation, and computing the entropy of TFN $\tilde{C}$, the integration result is shown as

$$H(\tilde{C})=\int_{a_1-b_3}^{a_2-b_2}\left[-(y)\ln(y)-(1-y)\ln(1-y)\right]p(x)dx+\int_{a_2-b_2}^{a_3-b_1}\left[-(z)\ln(z)-(1-z)\ln(1-z)\right]p(x)dx$$

$$\because \quad y=\frac{x-(a_1-b_3)}{(a_2-b_2)-(a_1-b_3)} \qquad \therefore \quad dx=((a_2-b_2)-(a_1-b_3))dy$$

$$\because \quad z=\frac{(a_3-b_1)-x}{(a_3-b_1)-(a_2-b_2)} \qquad \therefore \quad dx=-((a_3-b_1)-(a_2-b_2))dz$$

$$H(\tilde{C})=(-s)\left[(a_2-b_2)-(a_1-b_3)\right]\int_0^1\left[y\ln y+(1-y)\ln(1-y)\right]dy$$

$$+(s)\left[(a_3-b_1)-(a_2-b_2)\right]\int_0^1\left[z\ln z+(1-z)\ln(1-z)\right]dz$$

$$=(-s)\left[(a_2-b_2)-(a_1-b_3)\right]\left(\ln 1-\frac{1}{2}\right)+(s)\left[(a_3-b_1)-(a_2-b_2)\right]\left(\ln 1-\frac{1}{2}\right)$$

$$=\frac{s}{2}\left(2(a_2-b_2)-(a_1-b_3)-(a_3-b_1)\right)=\frac{s}{2}\left(2a_2-a_1-a_3\right)-\frac{s}{2}\left(2b_2-b_1-b_3\right)$$

$$=H(\tilde{A})-H(\tilde{B})$$

Hence, $H(\tilde{C})=\frac{s}{2}\left(2a_2-a_1-a_3\right)-\frac{s}{2}\left(2b_2-b_1-b_3\right)=H(\tilde{A})-H(\tilde{B})$.

Theorem 4. *Supposing* $\tilde{C} = 2\tilde{A}$ *,* $\tilde{C}$ *being a TFN, then the entropy of* $\tilde{C}$ *is twice the entropy of* $\tilde{A}$ *. That is,* $H(\tilde{C}) = H(2\tilde{A}) = 2H(\tilde{A})$.

Proof. For $\tilde{C}(x) = 2\tilde{A}(x)$, the membership function can be expressed as

$$\mu_{\tilde{C}}(x) = \begin{cases} 0 \quad , & x < 2a_1,\ 2a_3 < x \\ \dfrac{x - 2a_1}{2a_2 - 2a_1}, & 2a_1 \le x \le 2a_2 \\ \dfrac{2a_3 - x}{2a_3 - 2a_2}, & 2a_2 \le x \le 2a_3 \end{cases}$$

Let $y = \dfrac{x - 2a_1}{2a_2 - 2a_1}$, $z = \dfrac{2a_3 - x}{2a_3 - 2a_2}$.

Applying Eq.(4) to the above equation, we obtain

$$h(\mu_{\tilde{C}}(x)) = \begin{cases} 0 \quad , & x < 2a_1,\ 2a_3 < x \\ -(y)\ln(y) - (1-y)\ln(1-y), & 2a_1 \le x \le 2a_2 \\ -(z)\ln(z) - (1-z)\ln(1-z), & 2a_2 \le x \le 2a_3 \end{cases}$$

Applying Eq.(1) to the above equation, and computing the entropy of TFN $\tilde{C}$, the integration result is shown as

$$H(\tilde{C}) = \int_{2a_1}^{2a_2} \left[-(y)\ln(y) - (1-y)\ln(1-y)\right] p(x)dx + \int_{2a_2}^{2a_3} \left[-(z)\ln(z) - (1-z)\ln(1-z)\right] p(x)dx$$

$$\because \quad y = \frac{x - 2a_1}{2a_2 - 2a_1} \quad \therefore \ dx = (2a_2 - 2a_1)dy, \quad \because \ z = \frac{2a_3 - x}{2a_3 - 2a_2} \quad \therefore \ dx = -(2a_3 - 2a_2)dz$$

$$\begin{aligned} H(\tilde{C}) &= (-s)(2a_2 - 2a_1)\int_0^1 \left[y\ln y + (1-y)\ln(1-y)\right]dy + (s)(2a_3 - 2a_2)\int_0^1 \left[z\ln z + (1-z)\ln(1-z)\right]dz \\ &= (-s)(2a_2 - 2a_1)\left(\ln 1 - \frac{1}{2}\right) + (s)(2a_3 - 2a_2)\left(\ln 1 - \frac{1}{2}\right) \\ &= \frac{s}{2}\left(2 \times 2a_2 - 2a_1 - 2a_3\right) = 2 \times \frac{s}{2}\left(2a_2 - a_1 - a_3\right) \\ &= 2H(\tilde{A}) \end{aligned}$$

Hence, $H(\tilde{C}) = 2 \times \dfrac{s}{2}\left(2a_2 - a_1 - a_3\right) = 2H(\tilde{A})$.

Theorem 5. *Supposing* $\tilde{C} = -2\tilde{A}$ *,* $\tilde{C}$ *being a TFN, then the entropy of* $\tilde{C}$ *is negative two times the entropy of* $\tilde{A}$ *. That is,* $H(\tilde{C}) = H(-2\tilde{A}) = (-2)H(\tilde{A})$.

Proof. For $\tilde{C}(x) = -2\tilde{A}(x)$, the membership function can be expressed as,

$$\mu_{\tilde{C}}(x) = \begin{cases} 0 \quad , & x < -2a_3,\ -2a_1 < x \\ \dfrac{x + 2a_3}{2a_3 - 2a_2}, & -2a_3 \le x \le -2a_2 \\ \dfrac{-2a_1 - x}{2a_2 - 2a_1}, & -2a_2 \le x \le -2a_1 \end{cases}$$

Let $y = \dfrac{x + 2a_3}{2a_3 - 2a_2}$, $z = \dfrac{-2a_1 - x}{2a_2 - 2a_1}$

Applying Eq.(4) to the above equation, we obtain

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$$h(\mu_{\tilde{C}}(x))=\begin{cases}0, & x<-2a_3,\ -2a_1<x\\ -(y)\ln(y)-(1-y)\ln(1-y), & -2a_3\le x\le -2a_2\\ -(z)\ln(z)-(1-z)\ln(1-z), & -2a_2\le x\le -2a_1\end{cases}$$

Applying Eq.(1) to the above equation, and computing the entropy of TFN $\tilde{C}$, the integration result is shown as

$$H(\tilde{C})=\int_{-2a_3}^{-2a_2}\left[-(y)\ln(y)-(1-y)\ln(1-y)\right]p(x)dx+\int_{-2a_2}^{-2a_1}\left[-(z)\ln(z)-(1-z)\ln(1-z)\right]p(x)dx$$

$$\because\ y=\frac{x+2a_3}{2a_3-2a_2}\quad \therefore\ dx=(2a_3-2a_2)dy,\quad \because\ z=\frac{-2a_1-x}{2a_2-2a_1}\quad \therefore\ dx=-(2a_2-2a_1)dz$$

$$\begin{aligned}H(\tilde{C})&=(-s)(2a_3-2a_2)\int_0^1\left[y\ln y+(1-y)\ln(1-y)\right]dy+(s)(2a_2-2a_1)\int_0^1\left[z\ln z+(1-z)\ln(1-z)\right]dz\\ &=(-s)(2a_3-2a_2)\left(\ln 1-\frac{1}{2}\right)+(s)(2a_2-2a_1)\left(\ln 1-\frac{1}{2}\right)\\ &=\frac{s}{2}\left[(-2)\times 2a_2+2a_1+2a_3\right]=(-2)\times\frac{s}{2}\left(2a_2-a_1-a_3\right)\\ &=(-2)H(\tilde{A})\end{aligned}$$

Hence, $H(\tilde{C})=(-2)\times\frac{s}{2}\left(2a_2-a_1-a_3\right)=(-2)H(\tilde{A})$.

Remark 1. *combining Theorems 4 and 5, the result is that, if the "base width" of the TFN $\tilde{C}$ is positive (or negative) two times the "base width" of $\tilde{A}$, then the entropy of $\tilde{C}$ is positive (or negative) two times the entropy of $\tilde{A}$, using Shannon's Function to compute their entropies.*

Theorem 6. *Supposing $\tilde{A}^-$ is the image of TFN $\tilde{A}$, then the entropy of $\tilde{A}^-$ is equal to the negative entropy of $\tilde{A}$. That is, $H(\tilde{A}^-)=H(-\tilde{A})=-H(\tilde{A})$.*

Proof. For $\tilde{A}^-(x)=-\tilde{A}(x)$, the membership function can be expressed as

$$\mu_{\tilde{A}^-}(x)=\begin{cases}0, & x<-a_3,\ -a_1<x\\ \dfrac{x+a_3}{a_3-a_2}, & -a_3\le x\le -a_2\\ \dfrac{-a_1-x}{a_2-a_1}, & -a_2\le x\le -a_1\end{cases}$$

Let $y=\dfrac{x+a_3}{a_3-a_2}$, $z=\dfrac{-a_1-x}{a_2-a_1}$

Applying Eq.(4) to the above equation, we obtain

$$h(\mu_{\tilde{A}^-}(x))=\begin{cases}0, & x<-a_3,\ -a_1<x\\ -(y)\ln(y)-(1-y)\ln(1-y), & -a_3\le x\le -a_2\\ -(z)\ln(z)-(1-z)\ln(1-z), & -a_2\le x\le -a_1\end{cases}$$

Applying Eq.(1) to the above equation, and computing the entropy of TFN $\tilde{C}$, the integration result is shown as

$$H(\tilde{A}^-)=\int_{-a_3}^{-a_2}\left[-(y)\ln(y)-(1-y)\ln(1-y)\right]p(x)dx+\int_{-a_2}^{-a_1}\left[-(z)\ln(z)-(1-z)\ln(1-z)\right]p(x)dx$$

$$\because\ y=\frac{x+a_3}{a_3-a_2}\quad \therefore\ dx=(a_3-a_2)dy,\quad \because\ z=\frac{-a_1-x}{a_2-a_1}\quad \therefore\ dx=-(a_2-a_1)dz$$

$$H(\tilde{A}^{-})=(-s)(a_3-a_2)\int_0^1\left[y\ln y+(1-y)\ln(1-y)\right]dy+(s)(a_2-a_1)\int_0^1\left[z\ln z+(1-z)\ln(1-z)\right]dz$$
$$=(-s)(a_3-a_2)\left(\ln 1-\frac{1}{2}\right)+(s)(a_2-a_1)\left(\ln 1-\frac{1}{2}\right)$$
$$=\frac{s}{2}\left[(-1)\times 2a_2+a_1+a_3\right]=(-1)\times\frac{s}{2}\left(2a_2-a_1-a_3\right)$$
$$=(-1)H(\tilde{A})$$

Hence, $H(\tilde{A}^{-})=(-1)\times\frac{s}{2}\left(2a_2-a_1-a_3\right)=-H(\tilde{A})$.

Remark 2. *Supposing $\tilde{A}^{-}$ is the image of TFN $\tilde{A}$, then the entropy of $\tilde{A}^{-}$ is equal to the negative entropy of $\tilde{A}$, using Shannon's Function to compute their entropies. It can be expressed as* $H(\tilde{A}^{-})+H(\tilde{A})=0$.

Theorem 7. *Supposing $\tilde{C}=n\tilde{A},\ \ n\in N$, $\tilde{C}$ being a TFN, then the entropy of $\tilde{C}$ is n times the entropy of $\tilde{A}$. That is,* $H(\tilde{C})=H(n\tilde{A})=nH(\tilde{A})$.

Proof. $H(\square)=n\times\frac{s}{2}\left(2a_2-a_1-a_3\right),\ \ n\in N$.

$n=1$, $H(\square)=\frac{s}{2}\left(2a_2-a_1-a_3\right)=H(\tilde{A})$, is established. Suppose $n=k$ is established,

$H(\square)=k\frac{s}{2}\left(2a_2-a_1-a_3\right)=kH(\tilde{A})$.

Then when $n=k+1$,

$$H(\square)=(k+1)\frac{s}{2}\left(2a_2-a_1-a_3\right)=k\frac{s}{2}\left(2a_2-a_1-a_3\right)+\frac{s}{2}\left(2a_2-a_1-a_3\right)=(k)H(\tilde{A})+H(\tilde{A})$$

is established. By using mathematical induction, we can prove that when $\tilde{C}=n\tilde{A},\ \ n\in N$, then $H(\tilde{C})=nH(\tilde{A})$ is established.

Remark 3. *If the "base width" of the TFN $\tilde{C}$ is n $(n\in N)$ times the "base width" of the TFN $\tilde{A}$, then the entropy of $\tilde{C}$ is n times the entropy of $\tilde{A}$, using Shannon's Function to compute their entropies.*

Theorem 8. *Supposing $\tilde{C}=-n\tilde{A},\ \ n\in N$, $\tilde{C}$ being a TFN, then the entropy of $\tilde{C}$ is n times the entropy of $\tilde{A}$. That is,* $H(\tilde{C})=H(-n\tilde{A})=(-n)H(\tilde{A})$.

Proof. For $\tilde{C}(x)=-n\tilde{A}(x)$, the membership function can be expressed as

$$\mu_{\tilde{C}}(x)=\begin{cases}0 \quad , & x<-na_3,\ \ -na_1<x\\ \dfrac{x+na_3}{na_3-na_2}, & -na_3\le x\le -na_2\\ \dfrac{-na_1-x}{na_2-na_1}, & -na_2\le x\le -na_1\end{cases}$$

Let $y=\dfrac{x+na_3}{na_3-na_2}$, $z=\dfrac{-na_1-x}{na_2-na_1}$

Applying Eq.(4) to the above equation, we obtain

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$$h(\mu_{\widetilde{C}}(x))=\begin{cases} 0 & , \quad x<-na_3,\ -na_1<x \\ -(y)\ln(y)-(1-y)\ln(1-y), & -na_3\le x\le -na_2 \\ -(z)\ln(z)-(1-z)\ln(1-z), & -na_2\le x\le -na_1 \end{cases}$$

Applying Eq.(1) to the above equation, and computing the entropy of TFN $\widetilde{C}$, the integration result is shown as

$$H(\widetilde{C})=\int_{-na_3}^{-na_2}\left[-(y)\ln(y)-(1-y)\ln(1-y)\right]p(x)dx+\int_{-na_2}^{-na_1}\left[-(z)\ln(z)-(1-z)\ln(1-z)\right]p(x)dx$$

$$\because\ \ y=\frac{x+na_3}{na_3-na_2}\qquad \therefore\ \ dx=(na_3-na_2)dy,\qquad \because\ \ z=\frac{-na_1-x}{na_2-na_1}\qquad \therefore\ \ dx=-(na_2-na_1)dz$$

$$\begin{aligned} H(\widetilde{C})&=(-s)(na_3-na_2)\int_0^1\left[y\ln y+(1-y)\ln(1-y)\right]dy+(s)(na_2-na_1)\int_0^1\left[z\ln z+(1-z)\ln(1-z)\right]dz \\ &=(-s)(na_3-na_2)\left(\ln 1-\frac{1}{2}\right)+(s)(na_2-na_1)\left(\ln 1-\frac{1}{2}\right) \\ &=\frac{s}{2}\left[(-n)\times 2a_2+na_1+na_3\right]=(-n)\times\frac{s}{2}\left(2a_2-a_1-a_3\right) \\ &=(-n)H(\widetilde{A}) \end{aligned}$$

Hence, $H(\widetilde{C})=H(-n\,\widetilde{A})=(-n)H(\widetilde{A})$.

Remark 4. *Combining Theorems 7 and 8, the result is for any integer* m, $m\in I$, $\widetilde{C}=m\,\widetilde{A}$, *then the following relationship is always true:* $H(\widetilde{C})=H(m\,\widetilde{A})=mH(\widetilde{A})$.

Theorem 9. *Supposing* $\widetilde{C}=(\widetilde{A}+\widetilde{B})/2$, $\widetilde{C}$ *being a TFN, then the entropy of* $\widetilde{C}$ *is the average of the summation of* $\widetilde{A}$ *and* $\widetilde{B}$. *That is,* $H(\widetilde{C})=H\left((\widetilde{A}+\widetilde{B})/2\right)=\dfrac{H(\widetilde{A})+H(\widetilde{B})}{2}$.

Proof. From Eqs. (8) and (9) we know that $H(\widetilde{A})=\frac{s}{2}(2a_2-a_1-a_3)$ and $H(\widetilde{B})=\frac{s}{2}(2b_2-b_1-b_3)$, then

$$\begin{aligned} H(\widetilde{C})&=\frac{s}{2}\left(\frac{2a_2+2b_2}{2}-\frac{a_1+b_1}{2}-\frac{a_3+b_3}{2}\right) \\ &=\frac{s}{2}\times\frac{1}{2}\left[\left(2a_2+2b_2\right)-\left(a_1+b_1\right)-\left(a_3+b_3\right)\right]=\frac{1}{2}\times\left[\frac{s\left(2a_2-a_1-a_3\right)}{2}+\frac{s\left(2b_2-b_1-b_3\right)}{2}\right] \\ &=\frac{H(\widetilde{A})+H(\widetilde{B})}{2}. \qquad \square \end{aligned}$$

Theorem 10. *Supposing* $\widetilde{C}=\frac{1}{n}\sum_{i=1}^{n}\widetilde{A}_i$ *and* $\widetilde{A}_1,\widetilde{A}_2,\dots,\widetilde{A}_n$ *are TFNs, where* $\widetilde{A}_i=\left(a_1^i,\ a_2^i,\ a_3^i\right)$, $i=1,2,\dots,n$, $\widetilde{C}$ *also being a TFN, then* $H(\widetilde{C})=H\left(\frac{1}{n}\sum_{i=1}^{n}\widetilde{A}_i\right)=\frac{1}{n}\sum_{i=1}^{n}H(\widetilde{A}_i)$, *where* $n\ge 2$, $n\in N$.

Proof. From Eqs. (8) and (9) we know that $H(\widetilde{A}_i)=\frac{s}{2}\left(2a_2^i-a_1^i-a_3^i\right)$, $i=1,2,\dots,n$, $i\in N$, then

$$\begin{aligned} H(\widetilde{C})&=\frac{s}{2}\left[\frac{2a_2^1+2a_2^2+\dots+2a_2^n}{n}-\frac{a_1^1+a_1^2+\dots+a_1^n}{n}-\frac{a_3^1+a_3^2+\dots+a_3^n}{n}\right] \\ &=\frac{s}{2}\times\frac{1}{n}\left[\left(2a_2^1+2a_2^2+\dots+2a_2^n\right)-\left(a_1^1+a_1^2+\dots+a_1^n\right)-\left(a_3^1+a_3^2+\dots+a_3^n\right)\right] \end{aligned}$$

$$=\frac{1}{n}\times\left[\frac{s\left(2a_2^1-a_1^1-a_3^1\right)}{2}+\frac{s\left(2a_2^2-a_1^2-a_3^2\right)}{2}+\ldots+\frac{s\left(2a_2^n-a_1^n-a_3^n\right)}{2}\right]$$
$$=\frac{1}{n}\times\left[H(\tilde{A}_1)+H(\tilde{A}_2)+\ldots+H(\tilde{A}_n)\right]=\frac{1}{n}\sum_{i=1}^{n}H(\tilde{A}_i)\cdot \quad \square$$

Theorem 11. *Supposing* $\tilde{C}=\sum_{i=1}^{n}\tilde{A}_i$ *and* $\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n$ *are TFNs,* $\tilde{C}$ *also being a TFN, then the entropy of* $\tilde{C}$ *is the summation of the entropy of each TFN. That is,* $H(\tilde{C})=H\left(\sum_{i=1}^{n}\tilde{A}_i\right)=\sum_{i=1}^{n}H(\tilde{A}_i)$, $n\geq 2$, $n\in N$.

Proof. From Theorem 2 we know that if $\tilde{C}=\tilde{A}_1+\tilde{A}_2$, then $H(\tilde{C})=H(\tilde{A}_1)+H(\tilde{A}_2)$. Hence, it can be written as

$$\tilde{C}=\tilde{A}_{k+1}+\tilde{A}_{k+2}\Rightarrow H(\tilde{C})=H(\tilde{A}_{k+1})+H(\tilde{A}_{k+2}) \tag{A}$$

where $k\geq 0$, $k\in N$. This theorem proves that

$n=2$, $\tilde{C}=\tilde{A}_1+\tilde{A}_2\Rightarrow H(\tilde{C})=H(\tilde{A}_1)+H(\tilde{A}_2)$ holds.

Supposing $n=k$ holds, then

$$\tilde{C}=\tilde{A}_1+\tilde{A}_2+\ldots+\tilde{A}_k\Rightarrow H(\tilde{C})=H(\tilde{A}_1)+H(\tilde{A}_2)+\ldots+H(\tilde{A}_k) \tag{B}$$

when $n=k+2$, then $\tilde{C}=\tilde{A}_1+\tilde{A}_2+\ldots+\tilde{A}_k+\tilde{A}_{k+1}+\tilde{A}_{k+2}$.

From formulas (A) and (B)

$$\Rightarrow H(\tilde{C})=H(\tilde{A}_1)+H(\tilde{A}_2)+\ldots+H(\tilde{A}_n)+H(\tilde{A}_{k+1})+H(\tilde{A}_{k+2})$$

$\Rightarrow n=k+2$, it still holds. Hence, this theorem is true.

Theorem 12. *Suppose* $\tilde{C}$, $\tilde{A}$, *and* $\tilde{B}$ *are three TFNs, and they exit the linear relation as follows:* $\tilde{C}=n\,\tilde{A}+m\,\tilde{B}$, $n,m\in I$. *The entropies of* $\tilde{C}$, $\tilde{A}$, *and* $\tilde{B}$ *exit the relation as:* $H(\tilde{C})=H(n\,\tilde{A}+m\,\tilde{B})=nH(\tilde{A})+mH(\tilde{B})$.

Proof. We set $\tilde{C}_1=n\,\tilde{A}$ and $\tilde{C}_2=m\,\tilde{B}$. From Theorems 7 and 8, we know that

$$H(\tilde{C}_1)=nH(\tilde{A}),\ H(\tilde{C}_2)=mH(\tilde{B}) \tag{A}$$

We set $\tilde{C}=\tilde{C}_1+\tilde{C}_2$. From Theorem 2 we know that

$$H(\tilde{C})=H(\tilde{C}_1)+H(\tilde{C}_2) \tag{B}$$

From formulas (A) and (B), we obtain $H(\tilde{C})=nH(\tilde{A})+mH(\tilde{B})$ $\square$

Hence, the theorem is true.

Theorem 13. *Supposing* $\tilde{C}=r\,\tilde{A}$, $r\in R$, $\tilde{C}$ *being a TFN, then* $H(\tilde{C})=H(r\,\tilde{A})=rH(\tilde{A})$.

Proof. When $r\geq 0$, $\tilde{C}(x)=r\,\tilde{A}(x)$, then

$$\mu_{\tilde{C}}(x)=\begin{cases}0\quad, & x<ra_1, ra_3<x\\ \dfrac{x-ra_1}{ra_2-ra_1}, & ra_1\leq x\leq ra_2\\ \dfrac{ra_3-x}{ra_3-ra_2}, & ra_2\leq x\leq ra_3\end{cases}$$

Let $y=\dfrac{x-ra_1}{ra_2-ra_1}$, $z=\dfrac{ra_3-x}{ra_3-ra_2}$

Applying Eq.(4) to the above equation, we obtain

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$$h(\mu_{\tilde{C}}(x)) = \begin{cases} 0 & , \quad x < ra_1, ra_3 < x \\ -(y)\ln(y) - (1-y)\ln(1-y), & ra_1 \le x \le ra_2 \\ -(z)\ln(z) - (1-z)\ln(1-z), & ra_2 \le x \le ra_3 \end{cases}$$

Applying Eq.(1) to the above equation, and computing the entropy of TFN $\tilde{C}$, the integration result is shown as

$$H(\tilde{C}) = \int_{ra_1}^{ra_2} [-(y)\ln(y) - (1-y)\ln(1-y)]p(x)dx + \int_{ra_2}^{ra_3} [-(z)\ln(z) - (1-z)\ln(1-z)]p(x)dx$$

$$\because \quad y = \frac{x - ra_1}{ra_2 - ra_1} \quad \therefore \quad dx = (ra_2 - ra_1)dy, \quad \because \quad z = \frac{ra_3 - x}{ra_3 - ra_2} \quad \therefore \quad dx = -(ra_3 - ra_2)dz$$

$$H(\tilde{C}) = (-s)(ra_2 - ra_1)\int_0^1 [y\ln y + (1-y)\ln(1-y)]dy + (s)(ra_3 - ra_2)\int_0^1 [z\ln z + (1-z)\ln(1-z)]dz$$

$$= (-s)(ra_2 - ra_1)\left(\ln 1 - \frac{1}{2}\right) + (s)(ra_3 - ra_2)\left(\ln 1 - \frac{1}{2}\right)$$

$$= \frac{s}{2}(2ra_2 - ra_1 - ra_3)$$

$$= r \times \frac{s}{2}(2a_2 - a_1 - a_3)$$

$$= rH(\tilde{A})$$

Hence, $H(\tilde{C}) = r \times \frac{s}{2}(2a_2 - a_1 - a_3) = rH(\tilde{A})$.

When $r<0$, $H(\tilde{C}) = H(r\tilde{A}) = rH(\tilde{A})$ still holds. Hence, this theorem is true.

Theorem 14. *Suppose* $\tilde{C}$, $\tilde{A}$, *and* $\tilde{B}$ *are three TFNs, and they exit the linear relation as follows:* $\tilde{C} = r\tilde{A} + s\tilde{B}, \quad r, s \in R$. *The entropies of* $\tilde{C}$, $\tilde{A}$, *and* $\tilde{B}$ *exit the relation as:*

$H(\tilde{C}) = H(r\tilde{A} + s\tilde{B}) = rH(\tilde{A}) + sH(\tilde{B})$.

Proof. We set $\tilde{C}_1 = r\tilde{A}$ and $\tilde{C}_2 = s\tilde{B}$. From Theorem 13, we know that

$$H(\tilde{C}_1) = rH(\tilde{A}), \; H(\tilde{C}_2) = sH(\tilde{B}) \tag{A}$$

We set $\tilde{C} = \tilde{C}_1 + \tilde{C}_2$. From Theorem 2 we know that

$$H(\tilde{C}) = H(\tilde{C}_1) + H(\tilde{C}_2). \tag{B}$$

From formulas (A) and (B), we obtain $H(\tilde{C}) = rH(\tilde{A}) + sH(\tilde{B})$. □
Hence, the theorem is true.

Theorem 15. $\tilde{A}$ *is a TFN, expressed as* $\tilde{A} = (a_1, a_2, a_3)$. *Suppose* $2a_2 = a_1 + a_3$, *the TFN* $\tilde{A}$ *is an* isosceles triangle, *then the entropy of* $\tilde{A}$ *is zero (0s). That is,* $H(\tilde{A}) = 0s$.

Proof. From Theorem 1, the entropy of $\tilde{A}$ expressed as, $H(\tilde{A}) = \frac{s}{2}(2a_2 - a_1 - a_3)$, *the TFN* $\tilde{A}$ *is an* isosceles triangle, so $2a_2 = a_1 + a_3$, then $H(\tilde{A}) = \frac{s}{2}(2a_2 - a_1 - a_3) = 0s$.

4. Examples

In this section, we use real numbers to illustrate the above theorems. Suppose that there are three triangular fuzzy numbers $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{D}$ with real digits, briefly indicated as $\widetilde{A}=(-1,3,5)$, $\widetilde{B}=(0,2,6)$ and $\widetilde{D}=(1,4.5,8)$, respectively. Their membership functions and figures are shown below:

$$\mu_{\widetilde{A}}(x)=\begin{cases} 0 \quad , & x<-1,\ 5<x \\ \dfrac{x-(-1)}{3-(-1)}, & -1\le x\le 3 \\ \dfrac{5-x}{5-3}, & 3\le x\le 5 \end{cases} \tag{10}$$

$$\mu_{\widetilde{B}}(x)=\begin{cases} 0 \quad , & x<0,\ 6<x \\ \dfrac{x-0}{2-0}, & 0\le x\le 2 \\ \dfrac{6-x}{6-2}, & 2\le x\le 6 \end{cases} \tag{11}$$

$$\mu_{\widetilde{D}}(x)=\begin{cases} 0 \quad , & x<1,\ 8<x \\ \dfrac{x-1}{4.5-1}, & 1\le x\le 4.5 \\ \dfrac{8-x}{8-4.5}, & 4.5\le x\le 8 \end{cases} \tag{12}$$

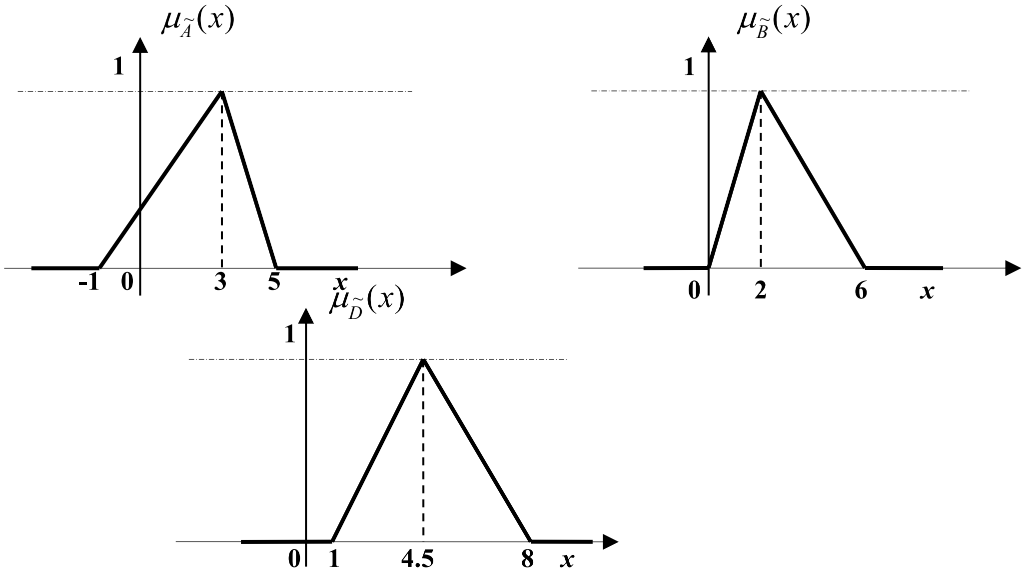

Figure 2: Three TFNs $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{D}$ with real digits

Example 1. Suppose there are two TFNs $\widetilde{A}$ and $\widetilde{B}$ whose membership functions are expressed as Eqs.(10) and (11), respectively. Thus, the following relationship holds: $H(\widetilde{A})=(-1)H(\widetilde{B})$.

Sol. Applying Eqs. (4) and (1) to Eqs. (10) and (11), respectively, the integration result is shown as

$$\begin{aligned} H(\widetilde{A}) &= \int_{-1}^{3}\left[-\left(\frac{x-(-1)}{3-(-1)}\right)\ln\left(\frac{x-(-1)}{3-(-1)}\right)-\left(1-\frac{x-(-1)}{3-(-1)}\right)\ln\left(1-\frac{x-(-1)}{3-(-1)}\right)\right]p(x)\,dx \\ &\quad +\int_{3}^{5}\left[-\left(\frac{5-x}{5-3}\right)\ln\left(\frac{5-x}{5-3}\right)-\left(1-\frac{5-x}{5-3}\right)\ln\left(1-\frac{5-x}{5-3}\right)\right]p(x)\,dx \\ &= s \end{aligned} \tag{13}$$

$$\begin{aligned} H(\widetilde{B}) &= \int_{0}^{2}\left[-\left(\frac{x-0}{2-0}\right)\ln\left(\frac{x-0}{2-0}\right)-\left(1-\frac{x-0}{2-0}\right)\ln\left(1-\frac{x-0}{2-0}\right)\right]p(x)\,dx \\ &\quad +\int_{2}^{6}\left[-\left(\frac{6-x}{6-2}\right)\ln\left(\frac{6-x}{6-2}\right)-\left(1-\frac{6-x}{6-2}\right)\ln\left(1-\frac{6-x}{6-2}\right)\right]p(x)\,dx \\ &= -s \end{aligned} \tag{14}$$

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$\therefore\ \dfrac{H(\tilde{A})}{H(\tilde{B})} = \dfrac{s}{(-s)} = -1 = \dfrac{2\times 3-(-1)-5}{2\times 2-0-6}$, the result is the ratio value of two TFNs $\tilde{A}$ and $\tilde{B}$ on the "*base width*". Thus, $H(\tilde{A}) = (-1)H(\tilde{B})$, where $k = -1$. Hence, Theorem 1 is true.

Example 2. Supposing TFN $\tilde{C}$ is the summation of TFNs $\tilde{A}$ and $\tilde{B}$, expressed as $\tilde{C} = \tilde{A} + \tilde{B}$, that is $\tilde{C} = (-1,5,11)$, then $H(\tilde{C}) = H(\tilde{A}) + H(\tilde{B})$.

Sol. The membership function of TFN $\tilde{C}$ is expressed as

$$\mu_{\tilde{C}}(x) = \begin{cases} 0 \quad , & x < -1,\ 11 < x \\ \dfrac{x-(-1)}{5-(-1)}, & -1 \le x \le 5 \\ \dfrac{11-x}{11-5}, & 5 \le x \le 11 \end{cases}$$

The entropy of TFN $\tilde{C}$ is calculated as the following:

$$\begin{aligned} H(\tilde{C}) &= \int_{-1}^{5}\left[-\left(\frac{x-(-1)}{5-(-1)}\right)\ln\left(\frac{x-(-1)}{5-(-1)}\right) - \left(1-\frac{x-(-1)}{5-(-1)}\right)\ln\left(1-\frac{x-(-1)}{5-(-1)}\right)\right]p(x)\,dx \\ &+ \int_{5}^{11}\left[-\left(\frac{11-x}{11-5}\right)\ln\left(\frac{11-x}{11-5}\right) - \left(1-\frac{11-x}{11-5}\right)\ln\left(1-\frac{11-x}{11-5}\right)\right]p(x)\,dx \\ &= 0s \end{aligned}$$

From Eqs.(13) and (14), we know $H(\tilde{A}) + H(\tilde{B}) = s + (-s) = 0s$.

Hence, $H(\tilde{C}) = H(\tilde{A} + \tilde{B}) = H(\tilde{A}) + H(\tilde{B})$. Thus, Theorem 2 is true.

Example 3. Supposing TFN $\tilde{C}$ is the difference of TFNs $\tilde{A}$ and $\tilde{B}$, expressed as $\tilde{C} = \tilde{A} - \tilde{B}$, that is $\tilde{C} = (-7,1,5)$, then $H(\tilde{C}) = H(\tilde{A}) - H(\tilde{B})$.

Sol. The membership function of TFN $\tilde{C}$ can be expressed as

$$\mu_{\tilde{C}}(x) = \begin{cases} 0 \quad , & x < -7,\ 5 < x \\ \dfrac{x-(-7)}{1-(-7)}, & -7 \le x \le 1 \\ \dfrac{5-x}{5-1}, & 1 \le x \le 5 \end{cases}$$

The entropy of TFN $\tilde{C}$ is calculated as the following:

$$\begin{aligned} H(\tilde{C}) &= \int_{-7}^{1}\left[-\left(\frac{x-(-7)}{1-(-7)}\right)\ln\left(\frac{x-(-7)}{1-(-7)}\right) - \left(1-\frac{x-(-7)}{1-(-7)}\right)\ln\left(1-\frac{x-(-7)}{1-(-7)}\right)\right]p(x)\,dx \\ &+ \int_{1}^{5}\left[-\left(\frac{5-x}{5-1}\right)\ln\left(\frac{5-x}{5-1}\right) - \left(1-\frac{5-x}{5-1}\right)\ln\left(1-\frac{5-x}{5-1}\right)\right]p(x)\,dx \\ &= 2s \end{aligned}$$

From Eqs.(13) and (14), we know $H(\tilde{A}) - H(\tilde{B}) = s - (-s) = 2s$.

Hence, $H(\tilde{C}) = H(\tilde{A} - \tilde{B}) = H(\tilde{A}) - H(\tilde{B})$. Thus, Theorem 3 is true.

Example 4. Supposing TFN $\tilde{C}$ is twice TFN $\tilde{A}$, expressed as $\tilde{C} = 2\tilde{A}$, that is $\tilde{C} = (-2,6,10)$, then $H(\tilde{C}) = H(2\tilde{A}) = 2H(\tilde{A})$.

Sol. The membership function of TFN $\tilde{C}$ can be expressed as

$$\mu_{\tilde{C}}(x)=\begin{cases}0 \quad , & x<-2,\ 10<x \\ \dfrac{x-(-2)}{6-(-1)}, & -2\le x\le 6 \\ \dfrac{10-x}{10-6}, & 6\le x\le 10\end{cases}$$

The entropy of TFN $\tilde{C}$ is calculated as the following:

$$H(\tilde{C})=\int_{-2}^{6}\left[-\left(\frac{x-(-2)}{6-(-2)}\right)\ln\left(\frac{x-(-2)}{6-(-2)}\right)-\left(1-\frac{x-(-2)}{6-(-2)}\right)\ln\left(1-\frac{x-(-2)}{6-(-2)}\right)\right]p(x)dx$$
$$+\int_{6}^{10}\left[-\left(\frac{10-x}{10-6}\right)\ln\left(\frac{10-x}{10-6}\right)-\left(1-\frac{10-x}{10-6}\right)\ln\left(1-\frac{10-x}{10-6}\right)\right]p(x)dx$$
$$=2s$$

From Eq.(13), we know $H(\tilde{A})=s$. Hence, $H(\tilde{C})=H(2\tilde{A})=2H(\tilde{A})$. Thus, Theorem 4 is true.

Example 5. Supposing TFN $\tilde{C}$ equals negative two times TFN $\tilde{A}$, expressed as $\tilde{C}=-2\tilde{A}$, that is $\tilde{C}=(-10,-6,2)$, then $H(\tilde{C})=H(-2\tilde{A})=(-2)H(\tilde{A})$.

Sol. The membership function of TFN $\tilde{C}$ can be expressed as

$$\mu_{\tilde{C}}(x)=\begin{cases}0 \quad , & x<-10,\ 2<x \\ \dfrac{x-(-10)}{(-6)-(-10)}, & -10\le x\le -6 \\ \dfrac{2-x}{2-(-6)}, & -6\le x\le 2\end{cases}$$

The entropy of TFN $\tilde{C}$ is calculated as the following:

$$H(\tilde{C})=\int_{-10}^{-6}\left[-\left(\frac{x-(-10)}{(-6)-(-10)}\right)\ln\left(\frac{x-(-10)}{(-6)-(-10)}\right)-\left(1-\frac{x-(-10)}{(-6)-(-10)}\right)\ln\left(1-\frac{x-(-10)}{(-6)-(-10)}\right)\right]p(x)dx$$
$$+\int_{-6}^{2}\left[-\left(\frac{2-x}{2-(-6)}\right)\ln\left(\frac{2-x}{2-(-6)}\right)-\left(1-\frac{2-x}{2-(-6)}\right)\ln\left(1-\frac{2-x}{2-(-6)}\right)\right]p(x)dx$$
$$=-2s$$

From Eq.(13), we know $H(\tilde{A})=s$. Hence, $H(\tilde{C})=H(-2\tilde{A})=(-2)H(\tilde{A})$. Thus, Theorem 5 is true.

Example 6. Supposing $\tilde{A}^{-}$ is the image of TFN $\tilde{A}$, that is $\tilde{A}^{-}=(-5,-3,1)$, then $H(\tilde{A}^{-})=H(-\tilde{A})=-H(\tilde{A})$.

Sol. The membership function of $\tilde{A}^{-}(x)=-\tilde{A}(x)$ can be expressed as

$$\mu_{\tilde{A}^{-}}(x)=\begin{cases}0 \quad , & x<-5,1<x \\ \dfrac{x-(-5)}{(-3)-(-5)}, & -5\le x\le -3 \\ \dfrac{1-x}{1-(-3)}, & -3\le x\le 1\end{cases}$$

The entropy of TFN $\tilde{A}^{-}$ is calculated as the following:

$$H(\tilde{A}^{-})=\int_{-5}^{-3}\left[-\left(\frac{x-(-5)}{(-3)-(-5)}\right)\ln\left(\frac{x-(-5)}{(-3)-(-5)}\right)-\left(1-\frac{x-(-5)}{(-3)-(-5)}\right)\ln\left(1-\frac{x-(-5)}{(-3)-(-5)}\right)\right]p(x)dx$$
$$+\int_{-3}^{1}\left[-\left(\frac{1-x}{1-(-3)}\right)\ln\left(\frac{1-x}{1-(-3)}\right)-\left(1-\frac{1-x}{1-(-3)}\right)\ln\left(1-\frac{1-x}{1-(-3)}\right)\right]p(x)dx$$

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$$= -s$$

From Eq.(13), we know $H(\widetilde{A}) = s$. Hence, $H(\widetilde{A}^{-}) = H(-\widetilde{A}) = -H(\widetilde{A})$. Thus, Theorem 6 is true.

Example 7. Supposing TFN $\widetilde{C}$ is *five times* TFN $\widetilde{A}$, expressed as $\widetilde{C} = 5\widetilde{A}$, that is $\widetilde{C} = (-5, 15, 25)$, then $H(\widetilde{C}) = H(5\widetilde{A}) = 5H(\widetilde{A})$.

Sol. The membership function of TFN $\widetilde{C}$ can be expressed as

$$\mu_{\widetilde{C}}(x) = \begin{cases} 0 \quad , & x < -5,\ 25 < x \\ \dfrac{x-(-5)}{15-(-5)}, & -5 \le x \le 15 \\ \dfrac{25-x}{25-15}, & 15 \le x \le 25 \end{cases}$$

The entropy of TFN $\widetilde{C}$ is calculated as the following:

$$H(\widetilde{C}) = \int_{-5}^{15} \left[-\left(\frac{x-(-5)}{15-(-5)} \right) \ln\left(\frac{x-(-5)}{15-(-5)} \right) - \left(1 - \frac{x-(-5)}{15-(-5)} \right) \ln\left(1 - \frac{x-(-5)}{15-(-5)} \right) \right] p(x)\,dx$$
$$+ \int_{15}^{25} \left[-\left(\frac{25-x}{25-15} \right) \ln\left(\frac{25-x}{25-15} \right) - \left(1 - \frac{25-x}{25-15} \right) \ln\left(1 - \frac{25-x}{25-15} \right) \right] p(x)\,dx$$
$$= 5s$$

From Eq.(13), we know $H(\widetilde{A}) = s$. Hence, $H(\widetilde{C}) = H(5\widetilde{A}) = 5H(\widetilde{A})$. Thus, Theorem 7 is true.

Example 8. Supposing TFN $\widetilde{C}$ equals negative six times TFN $\widetilde{A}$, expressed as $\widetilde{C} = -6\widetilde{A}$, that is $\widetilde{C} = (-30, -18, 6)$, then $H(\widetilde{C}) = H(-6\widetilde{A}) = (-6)H(\widetilde{A})$.

Sol. The membership function of TFN $\widetilde{C}$ can be expressed as

$$\mu_{\widetilde{C}}(x) = \begin{cases} 0 \quad , & x < -30,\ 6 < x \\ \dfrac{x-(-30)}{(-18)-(-30)}, & -30 \le x \le -18 \\ \dfrac{6-x}{6-(-18)}, & -18 \le x \le 6 \end{cases}$$

The entropy of TFN $\widetilde{C}$ is calculated as the following:

$$H(\widetilde{C}) = \int_{-30}^{-18} \left[-\left(\frac{x-(-30)}{(-18)-(-30)} \right) \ln\left(\frac{x-(-30)}{(-18)-(-30)} \right) - \left(1 - \frac{x-(-30)}{(-18)-(-30)} \right) \ln\left(1 - \frac{x-(-30)}{(-18)-(-30)} \right) \right] p(x)\,dx$$
$$+ \int_{-18}^{6} \left[-\left(\frac{6-x}{6-(-18)} \right) \ln\left(\frac{6-x}{6-(-18)} \right) - \left(1 - \frac{6-x}{6-(-18)} \right) \ln\left(1 - \frac{6-x}{6-(-18)} \right) \right] p(x)\,dx$$
$$= -6s$$

From Eq.(13), we know $H(\widetilde{A}) = s$. Hence, $H(\widetilde{C}) = H(-6\widetilde{A}) = (-6)H(\widetilde{A})$. Thus, Theorem 8 is true.

Example 9. Supposing TFN $\widetilde{C}$ is the average of TFNs $\widetilde{A}$ and $\widetilde{B}$, expressed as $\widetilde{C} = (\widetilde{A} + \widetilde{B})\big/2$, that is $\widetilde{C} = (-1/2, 5/2, 11/2)$, then $H(\widetilde{C}) = H((\widetilde{A} + \widetilde{B})/2) = (H(\widetilde{A})) + H(\widetilde{B}))/2$.

Sol. The membership function of TFN $\widetilde{C}$ can be expressed as

$$\mu_{\tilde{C}}(x)=\begin{cases} 0 & , \quad x<-\frac{1}{2}, \frac{11}{2}<x \\ \frac{x-(-1/2)}{5/2-(-1/2)}, & -\frac{1}{2}\le x\le \frac{5}{2} \\ \frac{11/2-x}{11/2-5/2}, & \frac{5}{2}\le x\le \frac{11}{2} \end{cases}$$

The entropy of TFN $\tilde{C}$ is calculated as the following:

$$\begin{aligned} H(\tilde{C}) &= \int_{-\frac{1}{2}}^{\frac{5}{2}}\left[-\left(\frac{x-(-1/2)}{5/2-(-1/2)}\right)\ln\left(\frac{x-(-1/2)}{5/2-(-1/2)}\right)-\left(1-\frac{x-(-1/2)}{5/2-(-1/2)}\right)\ln\left(1-\frac{x-(-1/2)}{5/2-(-1/2)}\right)\right]p(x)dx \\ &+\int_{\frac{5}{2}}^{\frac{11}{2}}\left[-\left(\frac{11/2-x}{11/2-5/2}\right)\ln\left(\frac{11/2-x}{11/2-5/2}\right)-\left(1-\frac{11/2-x}{11/2-5/2}\right)\ln\left(1-\frac{11/2-x}{11/2-5/2}\right)\right]p(x)dx \\ &= 0s \end{aligned}$$

From Eqs.(13) and (14), we know $\frac{H(\tilde{A})+H(\tilde{B})}{2}=\frac{s+(-s)}{2}=0s$.

Hence, $H(\tilde{C})=H\left(\frac{\tilde{A}+\tilde{B}}{2}\right)=\frac{H(\tilde{A})+H(\tilde{B})}{2}$. Thus, Theorem 9 is true.

Example 10. Supposing TFN $\tilde{C}$ is the average of TFNs $\tilde{A}$, $\tilde{B}$ and $\tilde{D}$, expressed as $\tilde{C}=\left(\tilde{A}+\tilde{B}+\tilde{D}\right)/3$, that is $\tilde{C}=(0,19/6,19/3)$, then $H(\tilde{C})=\left(H(\tilde{A})+H(\tilde{B})+H(\tilde{D})\right)/3$.

Sol. The membership function of TFN $\tilde{C}$ can be expressed as

$$\mu_{\tilde{C}}(x)=\begin{cases} 0 & , \quad x<0, \ \frac{19}{3}<x \\ \frac{x-0}{19/6-0}, & 0\le x\le \frac{19}{6} \\ \frac{19/3-x}{19/3-19/6}, & \frac{19}{6}\le x\le \frac{19}{3} \end{cases}$$

The entropy of TFN $\tilde{C}$ is calculated as the following:

$$\begin{aligned} H(\tilde{C}) &= \int_{0}^{\frac{19}{6}}\left[-\left(\frac{x-0}{19/6-0}\right)\ln\left(\frac{x-0}{19/6-0}\right)-\left(1-\frac{x-0}{19/6-0}\right)\ln\left(1-\frac{x-0}{19/6-0}\right)\right]p(x)dx \\ &+\int_{\frac{19}{6}}^{\frac{19}{3}}\left[-\left(\frac{19/3-x}{19/3-19/6}\right)\ln\left(\frac{19/3-x}{19/3-19/6}\right)-\left(1-\frac{19/3-x}{19/3-19/6}\right)\ln\left(1-\frac{19/3-x}{19/3-19/6}\right)\right]p(x)dx \\ &= 0s \end{aligned}$$

From Eqs.(13) and (14) we know that $\frac{H(\tilde{A})+H(\tilde{B})+H(\tilde{D})}{3}=\frac{s+(-s)+(0s)}{3}=0s$. Hence,

$H(\tilde{C})=H\left((\tilde{A}+\tilde{B}+\tilde{D})/3\right)=\left(H(\tilde{A})+H(\tilde{B})+H(\tilde{D})\right)/3$. Thus, Theorem 10 is true.

Example 11. Supposing TFN $\tilde{C}$ is the summation of TFNs $\tilde{A}$, $\tilde{B}$ and $\tilde{D}$, expressed as $\tilde{C}=\tilde{A}+\tilde{B}+\tilde{D}$, that is $\tilde{C}=(0,19/2,19)$, then $H(\tilde{C})=H(\tilde{A})+H(\tilde{B})+H(\tilde{D})$.

Sol. The membership function of TFN $\tilde{C}$ can be expressed as

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$$\mu_{\tilde{C}}(x)=\begin{cases}0 \quad, & x<0,\ 19<x\\ \dfrac{x-0}{19/2-0}, & 0\le x\le \dfrac{19}{2}\\ \dfrac{19-x}{19-19/2}, & \dfrac{19}{2}\le x\le 19\end{cases}$$

The entropy of TFN $\tilde{C}$ is calculated as the following:

$$H(\tilde{C})=\int_0^{\frac{19}{2}}\left[-\left(\frac{x-0}{19/2-0}\right)\ln\left(\frac{x-0}{19/2-0}\right)-\left(1-\frac{x-0}{19/2-0}\right)\ln\left(1-\frac{x-0}{19/2-0}\right)\right]p(x)dx$$
$$+\int_{\frac{19}{2}}^{19}\left[-\left(\frac{19-x}{19-19/2}\right)\ln\left(\frac{19-x}{19-19/2}\right)-\left(1-\frac{19-x}{19-19/2}\right)\ln\left(1-\frac{19-x}{19-19/2}\right)\right]p(x)dx$$
$$=0s$$

From Eqs.(13) and (14) we know that $H(\tilde{A})+H(\tilde{B})+H(\tilde{D})=s+(-s)+0s=0s$. Hence,

$H(\tilde{C})=H(\tilde{A}+\tilde{B}+\tilde{D})=H(\tilde{A})+H(\tilde{B})+H(\tilde{D})$. Thus, Theorem 11 is true.

Example 12. Supposing there are three TFNs $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3$, and they have the following relationship with TFNs $\tilde{A}$ and $\tilde{B}$, $\tilde{C}_1=5\tilde{A}+6\tilde{B}$, $\tilde{C}_2=(-5)\tilde{A}+6\tilde{B}$ and $\tilde{C}_3=(-5)\tilde{A}+(-6)\tilde{B}$. That is, $\tilde{C}_1=(-5,27,61)$, $\tilde{C}_2=(-25,-3,41)$ and $\tilde{C}_3=(-61,-27,5)$, then $H(\tilde{C}_1)=5H(\tilde{A})+6H(\tilde{B})$, $H(\tilde{C}_2)=(-5)H(\tilde{A})+6H(\tilde{B})$ and $H(\tilde{C}_3)=(-5)H(\tilde{A})+(-6)H(\tilde{B})$.

Sol. From Eqs.(13) and (14) we know that $H(\tilde{A})=s$ and $H(\tilde{B})=-s$. Now, we compute the entropies of $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3$ separately, as follows.

$$H(\tilde{C}_1)=\int_{-5}^{27}\left[-\left(\frac{x-(-5)}{27-(-5)}\right)\ln\left(\frac{x-(-5)}{27-(-5)}\right)-\left(1-\frac{x-(-5)}{27-(-5)}\right)\ln\left(1-\frac{x-(-5)}{27-(-5)}\right)\right]p(x)dx$$
$$+\int_{27}^{61}\left[-\left(\frac{61-x}{61-27}\right)\ln\left(\frac{61-x}{61-27}\right)-\left(1-\frac{61-x}{61-27}\right)\ln\left(1-\frac{61-x}{61-27}\right)\right]p(x)dx$$
$$=-s=5\times s+6\times(-s)=5H(\tilde{A})+6H(\tilde{B})$$

So $H(\tilde{C}_1)=H(5\tilde{A}+6\tilde{B})=5H(\tilde{A})+6H(\tilde{B})$ is true.

$$H(\tilde{C}_2)=\int_{-25}^{-3}\left[-\left(\frac{x-(-25)}{(-3)-(-25)}\right)\ln\left(\frac{x-(-25)}{(-3)-(-25)}\right)-\left(1-\frac{x-(-25)}{(-3)-(-25)}\right)\ln\left(1-\frac{x-(-25)}{(-3)-(-25)}\right)\right]p(x)dx$$
$$+\int_{-3}^{41}\left[-\left(\frac{41-x}{41-(-3)}\right)\ln\left(\frac{41-x}{41-(-3)}\right)-\left(1-\frac{41-x}{41-(-3)}\right)\ln\left(1-\frac{41-x}{41-(-3)}\right)\right]p(x)dx$$
$$=-11s=(-5)\times s+6\times(-s)=(-5)H(\tilde{A})+6H(\tilde{B})$$

So $H(\tilde{C}_2)=H(-5\tilde{A}+6\tilde{B})=(-5)H(\tilde{A})+6H(\tilde{B})$ is true.

$$H(\tilde{C}_3)=\int_{-61}^{-27}\left[-\left(\frac{x-(-61)}{(-27)-(-61)}\right)\ln\left(\frac{x-(-61)}{(-27)-(-61)}\right)-\left(1-\frac{x-(-61)}{(-27)-(-61)}\right)\ln\left(1-\frac{x-(-61)}{(-27)-(-61)}\right)\right]p(x)dx$$
$$+\int_{-27}^{5}\left[-\left(\frac{5-x}{5-(-27)}\right)\ln\left(\frac{5-x}{5-(-27)}\right)-\left(1-\frac{5-x}{5-(-27)}\right)\ln\left(1-\frac{5-x}{5-(-27)}\right)\right]p(x)dx$$
$$=s=(-5)\times s+(-6)\times(-s)=(-5)H(\tilde{A})+(-6)H(\tilde{B})$$

So $H(\tilde{C}_3)=H(-5\tilde{A}-6\tilde{B})=(-5)H(\tilde{A})+(-6)H(\tilde{B})$ is true. Hence Theorem 12 is true.

Example 13. Supposing there are two TFNs $\tilde{C}_1$ and $\tilde{C}_2$, and they have the following relationship with TFN $\tilde{A}$: $\tilde{C}_1 = 1.6\tilde{A}$ and $\tilde{C}_2 = (-3.8)\tilde{A}$. That is, $\tilde{C}_1 = (-1.6, 4.8, 8.0)$ and $\tilde{C}_2 = (-19.0, -11.4, 3.8)$, then $H(\tilde{C}_1) = 1.6H(\tilde{A})$, $H(\tilde{C}_2) = (-3.8)H(\tilde{A})$.

Sol. From Eq.(13), we know $H(\tilde{A}) = s$. Now, we compute the entropies of $\tilde{C}_1$ and $\tilde{C}_2$ separately, as follows:

$$H(\tilde{C}_1) = \int_{-1.6}^{4.8}\left[-\left(\frac{x-(-1.6)}{4.8-(-1.6)}\right)\ln\left(\frac{x-(-1.6)}{4.8-(-1.6)}\right)-\left(1-\frac{x-(-1.6)}{4.8-(-1.6)}\right)\ln\left(1-\frac{x-(-1.6)}{4.8-(-1.6)}\right)\right]p(x)\,dx$$
$$+\int_{4.8}^{8.0}\left[-\left(\frac{8.0-x}{8.0-4.8}\right)\ln\left(\frac{8.0-x}{8.0-4.8}\right)-\left(1-\frac{8.0-x}{8.0-4.8}\right)\ln\left(1-\frac{8.0-x}{8.0-4.8}\right)\right]p(x)\,dx$$
$$= 1.6s = 1.6\times s = 1.6H(\tilde{A})$$

Then $H(\tilde{C}_1) = H(1.6\tilde{A}) = 1.6H(\tilde{A})$ is true.

$$H(\tilde{C}_2) = \int_{-19.0}^{-11.4}\left[-\left(\frac{x-(-19.0)}{(-11.4)-(-19.0)}\right)\ln\left(\frac{x-(-19.0)}{(-11.4)-(-19.0)}\right)-\left(1-\frac{x-(-19.0)}{(-11.4)-(-19.0)}\right)\ln\left(1-\frac{x-(-19.0)}{(-11.4)-(-19.0)}\right)\right]p(x)\,dx$$
$$+\int_{-11.4}^{3.8}\left[-\left(\frac{3.8-x}{3.8-(-11.4)}\right)\ln\left(\frac{3.8-x}{3.8-(-11.4)}\right)-\left(1-\frac{3.8-x}{3.8-(-11.4)}\right)\ln\left(1-\frac{3.8-x}{3.8-(-11.4)}\right)\right]p(x)\,dx$$
$$= (-3.8)s = (-3.8)\times s = (-3.8)H(\tilde{A})$$

Then $H(\tilde{C}_2) = H(-3.8\tilde{A}) = (-3.8)H(\tilde{A})$ is true. Hence, Theorem 13 is true.

Example 14. Supposing TFN $\tilde{C}$ has the following relationship with TFNs $\tilde{A}$ and $\tilde{B}$: $\tilde{C} = 1.6\tilde{A} + (-2.3)\tilde{B}$. That is, $\tilde{C} = (-15.4, 0.2, 8)$, then $H(\tilde{C}) = H(1.6\tilde{A} + (-2.3)\tilde{B}) = (1.6)H(\tilde{A}) + (-2.3)H(\tilde{B})$.

Sol. The membership function of TFN $\tilde{C}$ can be expressed as

$$\mu_{\tilde{C}}(x) = \begin{cases} 0 \quad , & x < -15.4,\ 8 < x \\ \dfrac{x-(-15.4)}{0.2-(-15.4)}, & -15.4 \le x \le 0.2 \\ \dfrac{19-x}{8-0.2}, & 0.2 \le x \le 8 \end{cases}$$

The entropy of TFN $\tilde{C}$ is calculated as the following:

$$H(\tilde{C}) = \int_{-15.4}^{0.2}\left[-\left(\frac{x-(-15.4)}{0.2-(-15.4)}\right)\ln\left(\frac{x-(-15.4)}{0.2-(-15.4)}\right)-\left(1-\frac{x-(-15.4)}{0.2-(-15.4)}\right)\ln\left(1-\frac{x-(-15.4)}{0.2-(-15.4)}\right)\right]p(x)\,dx$$
$$+\int_{0.2}^{8}\left[-\left(\frac{19-x}{8-0.2}\right)\ln\left(\frac{19-x}{8-0.2}\right)-\left(1-\frac{19-x}{8-0.2}\right)\ln\left(1-\frac{19-x}{8-0.2}\right)\right]p(x)\,dx$$
$$= (3.9)s = (-3.8)\times s = (-3.8)H(\tilde{A})$$

From example 1, we know $H(\tilde{A}) = s$, $H(\tilde{B}) = -s$. From example 13, $H(1.6\tilde{A}) = 1.6s$,

$H(-2.3\tilde{B}) = (-2.3)\times(-s) = 2.3s$.

From above results, $H(\tilde{C}) = (3.9)s = 1.6s + 2.3s = H(1.6\tilde{A}) + H(-2.3\tilde{B}) = 1.6H(\tilde{A}) + (-2.3)H(\tilde{B})$ is true. Hence, Theorem 14 is true.

Example 15. The membership functions of the TFN $\tilde{D}$ is expressed as Eq.(12). the TFN $\tilde{D}$ is an isosceles triangle Thus, the entropy of $H(\tilde{D}) = 0s$.

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Sol. The membership function of TFN $\widetilde{D}$ can be expressed as Eq.(12), that is $\widetilde{D}=(a_1,a_2,a_3)=(1,4.5,8)$.

Because $2a_2=a_1+a_3,\ \ 2\times 4.5=1+8$, so TFN $\widetilde{D}$ is an isosceles triangle. From **Theorem 1**, we know $H(\widetilde{A})=\frac{s}{2}\left(2a_2-a_1-a_3\right)$, so the entropy of $\widetilde{D}$ is $H(\widetilde{D})=\frac{s}{2}\left(2\times 4.5-1-8\right)=0s$. Hence, Theorem 15 is true.

5. CONCLUSION

In this paper, we used Shannon's Function to find the entropy differences of triangular fuzzy numbers with the entropy of arithmetic operations and we summarized the above theorems and examples. The main conclusions are as follows.

Firstly, when two TFNs are added or subtracted, the entropy is equal to the addition or subtraction of the entropies of each individual TFN. Secondly, the entropy of the summation of many TFNs is equal to the summation of each of those TFNs' entropies. Thirdly, no matter whether two TFNs are left-facing or right-facing right triangles, if they have the same "base width", then their entropies are the same. Fourthly, if two TFNs are isosceles triangles, their entropies are the same. Fifthly, if TFNs are multiplied by positive integer numbers or positive real numbers, the entropy is positive. Sixthly, if TFNs are multiplied by negative integer numbers or negative real numbers, the entropy is negative. Seventhly, the entropy of a TFN is the same as the negative image of its TFN entropy. Eighthly, the entropy of the average of two TFNs is equal to the average of each of those two TFNs' entropies. Ninthly, the entropy of the average of many TFNs is equal to the average of those TFNs' entropies. Tenthly, if a positive or negative linear relationship exists between TFNs, a positive or negative linear relationship always exists between their entropies.

This paper proves that whether the entropies of triangular fuzzy numbers increase or decrease depends on the operations of arithmetic.

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